The Iterated Local Model for Social Networks

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Outline

Complex Networks

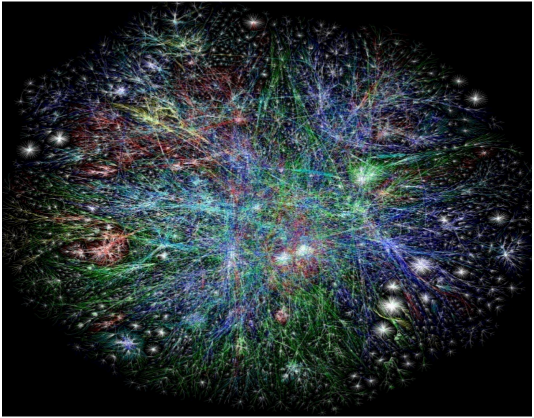

Figure: The Web Graph

Complex Networks

Four main properties:

1. Large-scale
2. Evolving over time
3. Power law degree distribution
4. Small world property

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2. Evolving over time
3. Power law degree distribution
4. Small world property
5. Densification

Power Law Degree Distribution

Degree Distribution: $\{N_{k,G} : 0 \leq k \leq n\}$

$$N_{k,G} = |\{x \in V(G) : \deg_G(x) = k|$$

Power Law: for $1 < \beta \in \mathbb{R}$, and interval of $k \in \mathbb{N}$

$$\frac{N_{k,G}}{n} \approx k^{-\beta}.$$

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Small World Property

The average distance is

$$L(G) = \frac{\sum_{u,v \in V(G)} d(u,v)}{\binom{|V(G)|}{2}}$$

The clustering coefficient of G is defined as follows:

$$C(G) = \frac{1}{|V(G)|} \sum_{x \in V(G)} C_x(G), \quad \text{where} \quad C_x(G) = \frac{\left|E\left(G[N_G(x)]\right)\right|}{\binom{\deg(x)}{2}}.$$

Small World Property

The average distance between any pairs of nodes should be low in graphs with the small world property, specifically,

$$L(G_t) = O(\log\log n)$$

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The clustering coefficient describes the density of edges within the neighbourset of a vertex, and should be large.

Densification

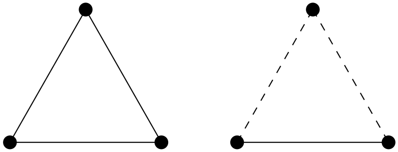
A sequence of graphs $\{G_t : t \in \mathbb{N}\}$ densifies over time if

$$\lim_{t \to \infty} \frac{|E(G_t)|}{|V(G_t)|} \to \infty$$

Outline

Structural Balance Theory

Representing adversarial relationships with $(-)$ and friendly relationships with $(+)$, Structural Balance Theory says triads seek a positive product of edge signs, called closure.



ILT

Iterated Local Transitivity Model (ILT) (2009, Bonato, Hadi, Horn, Prałat, Wang)

Input: G_0

To form G_t at time t clone each $x \in V(G_{t-1})$ by adding a new node x' such that

$$N_{G_t}(x') = N_{G_{t-1}}[x]$$

Example of ILT

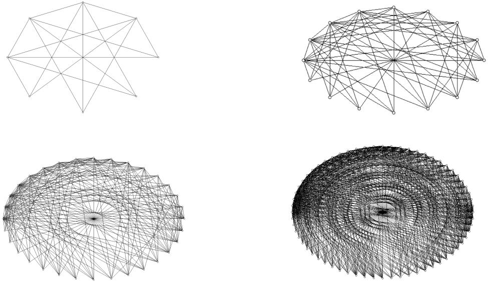


Figure: Example of ILT with $G_0 = C_4$.

Iterated Local Anti-Transitivity

Iterated Local Anti-Transitivity Model (ILAT) (2017, Bonato, Infeld, Pokhrel, Prałat)

Input: G_0

To form G_t at time t anti-clone each $x \in V(G_{t-1})$ by adding new node x^* such that

$$N_{G_t}(x^*) = V(G_{t-1}) \backslash N_{G_{t-1}}[x]$$

Example of ILAT

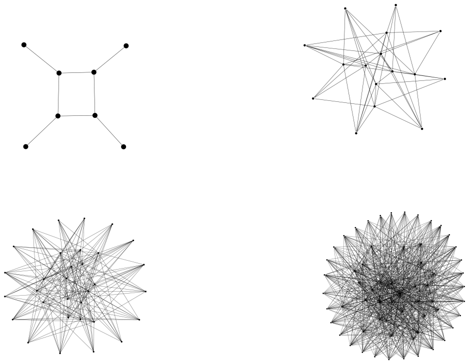


Figure: Example of ILAT with $G_0 = C_4$.

Outline

ILM

Iterated Local Model (ILM)

Input: G_0 and $S = \{b_i\}_{i \in \mathbb{N}}$, where $b_i \in \{0, 1\}$

To form $\mathrm{ILM}_{t,S}(G_0)$ at time t:

- if $b_t = 1$ add a clone x' for each $x \in V(G_{t-1})$ with

$$N_{G_t}(x') = N_{G_{t-1}}[x]$$

- if $b_t = 0$ add an anti-clone x^* for each $x \in V(G_{t-1})$ with

$$N_{G_t}(x^*) = V(G_{t-1}) \backslash N_{G_{t-1}}[x]$$

Example of ILM

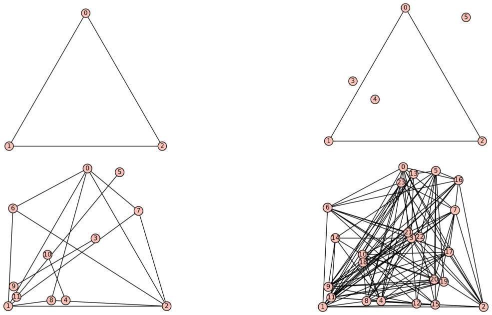


Figure: Example of ILM with $G_0 = K_3$ and $S = (0, 1, 0, 1, \ldots)$

Outline

Size, Evolution, and Densification

Theorem Given any graph, G_0, and any binary sequence, S, with at least one zero, then at time step t

$$|E(\mathrm{ILM}_{t,S}(G))| = \Theta\left(2^{t+\beta}\left(\frac{3}{2}\right)^{t-\beta}\right) = \Theta\left(2^{\beta}\left(\frac{3}{2}\right)^{t-\beta} n_t\right)$$

Where τ is the first index such that $s_\tau = 0$, and β is the largest index such that $s_\beta = 0$.

Size, Evolution and Densification

So far, ILM exhibits 3 of the 4+1 complex network properties

1. Large Scale
2. Evolving over time
5. Densification

Small WorldProperties

Theorem Given $G \neq K_1$ be a graph that is not the disjoint union of two cliques, and a sequence with at least two zeroes, then

$$\text{diam}(\text{ILM}_{t,S}(G)) = 3$$

Theorem Given a sequence with bounded gaps between zeroes, and k a constant such that there are no gaps of length k,

$$C(\text{ILM}_{t,S}(G)) \geq (1 + o(1))\frac{1}{2^{2k+4}}.$$

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The clustering coefficient is bounded away from zero.

Outline

Structural Results

Theorem If F is a graph, then there exists some constant $t_0 = t_0(F)$ such that for all $t \geq t_0$, all graphs G, and all binary sequences S, F is an induced subgraph of $\mathrm{ILM}_{t,S}(G)$

Other properties

- $\chi(G) + t - 1 \leq \chi\left(\mathrm{ILM}_{t,S}(G)\right) \leq \chi(G) + t$
- $\gamma(\mathrm{ILM}_{t,S}(G)) \leq 3$
- $\mathrm{ILM}_{t,S}(G)$ is hamiltonian

Outline

IGM

Iterated Global Model (IGM)

Input: G_0 and let $k \geq 1$ and integer

To form G_{t+1} from G_t at time t: for each set of vertices of cardinality $\lfloor \frac{1}{k} n_t \rfloor$, say S, add a new v_S that is adjacent to each vertex of S

The Half-Model

When $k = 2$ we call this The Half-Model.

We focus on this model and make use of Stirling's approximation for binomial coefficients

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}$$

Outline

Size, Evolution, and Densification

Theorem

$$n_t \sim \binom{n_{t-1}}{\left\lfloor \frac{n_{t-1}}{2} \right\rfloor} \qquad \text{and} \qquad e_t \sim \binom{n_{t-1}}{\left\lfloor \frac{n_{t-1}}{2} \right\rfloor} \cdot \left\lfloor \frac{n_{t-1}}{2} \right\rfloor$$

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Theorem The half-model densifies.

To simplify the notation we will use

$$\alpha_t = \binom{n_t}{\lfloor \frac{n_t}{2} \rfloor}$$

Complex Network Properties

Theorem Graphs generated by the half-model satisfy $\lambda_t \sim 1$, where λ_t is the spectral gap of G_t.

Theorem Given G_0 with at least 4 vertices, and $t \geq 4$ the the diameter of the half model is at most 3.

Outline

Structural Results

- The independence number of G_t is α_{t-1}
- $\omega(G_t) = \min\left(\left\lfloor \frac{n_{t-1}}{2} \right\rfloor + 1, \omega(G_0) + t\right)$
- $\chi(G_t) = \min\left(\chi(G_0) + t, \left\lfloor \frac{n_{t-1}}{2} \right\rfloor + 1\right)$
- The domination number of G_t is $\gamma(G_t) = \left\lceil \frac{n_{t-1}}{2} \right\rceil + 1$

Future Directions

- Graph Limits
- Clustering of IGM
- Cloning subgraphs
- Randomization of the models