MHF4U Final Exam Review

Writing your final exam:

- Show all your work. You will not receive full marks for a question if *insufficient work* is shown.
- Write out all solutions clearly, label all steps, and make note of final answers
- Attempt all questions. You can receive *partial marks* even if your solution is incomplete or your final answer is incorrect.
- Follow the recommended time for each section as much as possible

A. Unit 1 – Polynomial & Rational Functions

Topics at a glance:

(a) Polynomial Functions:

- end-behaviour, domain & range, turning points, max and min, finite differences
- symmetry: even & odd functions
- roots, order of a polynomial
- graphs of polynomials
- families of polynomials
- transformations
- division of polynomials
- remainder theorem
- factoring polynomials, factoring quadratics, sum and difference of cubes
- solving polynomial equations and inequalities (sign charts)

(b) Rational Functions:

- horizontal & vertical asymptotes, *x*- and *y*-intercepts, domain & range
- behaviour near vertical asymptotes
- transformations
- graphs of rational functions: $y=\dfrac{n}{ax+b}$, $y=\dfrac{ax+b}{cx+d}$, $y=\dfrac{n}{ax^2+bx+c}$ $n \in \mathbb{R}$
- special cases
- rational equations and inequalities (sign chart!)

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1. (a) Sketch the graphs of the following. (i) All *x*- and *y*-intercepts must be clearly identified on your sketch. (ii) Identify two additional points on your graph (ie. and *x* and *y* that satisfies *y=f(x)*).

(a) $f(x)=x^5-4x^3$ (b) $f(x)=(3x-1)^2(x+2)(7-2x)$

(c) $g(x)=-2x^3(x^2-1)$ (d) $y=x^4-15x^2-10x+24$

3. Using an <u>algebraic method</u> only, determine whether each of the following is an even function, an odd function, or neither.

(a) $y=7x^5-x^3+x^2+2$ (b) $y=(2x-1)^2(x-1)(3x-7)$

4. (a) Use the remainder theorem to determine the remainder when $2x^3-5x^2+x+2$ is divided by

4. (a) Use the remainder theorem to determine the remainder when $2x^3 - 5x^2 + x + 2$ is divided by $(2x - 1)$. Write your answer in <u>exact form</u>.

(b) For what value(s) of *k* will the polynomial $f(x) = x^3 + 2x^2 + kx + 5$ have the same remainder when it is divided by $x + 1$ and $x - 2$?

(c) The polynomial $mx^3 + 12x^2 - nx + 2$ has a remainder of 14 when divided by $x - 1$, and a remainder of 68 when divided by $x - 2$. What are the values of *m* and *n*?

5. Solve each equation by factoring completely.

(a) $x^3 - 7x + 6 = 0$

(b) $x^3(x - 1) = 11x^2 - 9(x + 2)$

(c) $x(x^2 - 4) = -3(x^2 - 4)$

6. (a) Divide $3x^3 - 5x^2 + 2x - 1$ by $(x - 2)$. Express the results in quotient form.

(b) Divide $2x^3 - 5x^2 + 2x - 7$ by $(x^2 + 1)$. Are there any restrictions on *x*? Explain.

7. The volume in cubic centimeters of a rectangular box is given by $2x^3 + x^2 - 27x - 36$. Find possible dimensions of the box if the height is given by $(x + 3)$ cm.

8. Find 2 consecutive even integers whose product is 5328.

9. (a) (i) Write the general equation for the family of polynomials having zeroes:
0 (order 2), –2 (order 1) and 3 (order 2).

(ii) Find the member of the family passing through the point (1, 3).

(iii) On the same axes, sketch graphs of any 3 members of this family.

(b) Find the general equation, in standard form, of the cubic having roots -3 and $2 \pm \sqrt{3}$.

10. (a) $y = x^5$ is vertically compressed by a factor of $\frac{2}{3}$, horizontally stretched by a factor of 2, reflected in both the *x*- and *y*-axes, translated 4 units to the left and $\frac{7}{2}$ units down. Write the equation of the transformed function *f*(*x*).
(b) Find *f*(–2). Write your answer in <u>exact form</u>.
(c) Find the <u>image</u> of (1, 1) on the base curve under the above transformations.

11. (a) Solve using sign chart: $2x^3 \geq 7x + 6 - x^2$.

(b) Solve using an algebraic method (by considering all possibilities): $(2x - 1)(x + 3)(5x - 3) \leq 0$.

(c) For what values of *x* does the graph of $y = x^2 - 7x$ lie below the graph of $y = 12 - 3x$?

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12. Sketch the graphs of the following. Each sketch must clearly identify all asymptotes and intercepts, and identify the domain and range for each.

(a) $y = \dfrac{3x-7}{2x+5}$ (b) $f(x) = \dfrac{-32}{x^2-16}$ (c) $f(x) = \dfrac{x+2}{x^2+9x+14}$

(e) $f(x) = \dfrac{x-3}{x+7} + 2$ (f) $y = \dfrac{x^2-9}{x^2-x-6}$. (g) $y = \dfrac{18}{x^2-9}$

13. The value of a car in dollars x years after it is bought is modelled by $f(x) = \dfrac{18000+5x}{2+0.5x} + 80$.

(a) Find the value of the car (i) when new, (ii) after 5 years.

(b) What is the least value the car can have? (c) sketch the graph of the function

14. Solve $\dfrac{x}{x-2} + \dfrac{1}{x+4} = \dfrac{2}{x^2-6x+8}$. Identify the restrictions on x.

15. Solve using a sign chart:

(a) $\dfrac{x^2-5x}{(x+3)(x-2)} > 0$ (b) $\dfrac{1}{2x+1} + \dfrac{1}{x+1} > \dfrac{8}{15}$.

16. John has a sister who is three years older than he is, and a brother who is two years younger than he is. How old must John be in order that the ratio of his sister's age to his brother's age is less than 2?

B. Unit 2 – Exponential & Logarithmic Functions

Topics at a glance:

- laws of logarithms
- exponential laws
- *key features & graphs of $y = \log_a x$ (asymptotes, intercepts, domain, range)
- *transformations of $y = \log x$; graphs of transformed functions
- inverse of exponential & logarithmic functions & graphical representation
- $y = \log_b(x) \Leftrightarrow x = b^y$
- equivalent exponential & logarithmic functions
- exponential and logarithmic equations
- applications of exponential and logarithmic functions

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1. Write in logarithmic form: (a) $2^{-3} = \frac{1}{8}$ (b) $5^{3k} = M$

2. Write in exponential form: (a) $\log \dfrac{1}{10000} = -4$ (b) $\log_a 3M = 2P$

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3. Evaluate the following without a calculator. (Full marks will not be awarded if insufficient work is shown.):

(a) $\log_5 4375 - \log_5 7$ (b) $\log_3 (81)^{-5}(\sqrt{27})$ (c) $\log_{\sqrt{5}} 125$ (e) $\log {}^{25}\!/_{2} + \log 80$

4. Write as a single logarithm and simplify completely:

(a) $3\log x^2 - 2\log y + \log mzx^4$

(b) $2 + \dfrac{\log x}{\log 3}$ (c) $\log(x^2 - 4) - \log(x - 2)$ and state the restrictions on x.

5. (a) What is the relationship between $\log_a b$ and $8\log_a \sqrt{b}$? Justify your answer.

(b) If $\log_a b^3 = 5$, what is the value of (a) $\log_b a^3$? (b) $\log_a \sqrt{b}$?

6. Solve: (a) $\log_2(3x - 5) = 3$ (b) $7^{3x+2} = 4^{x+5}$ (c) $\log_9(x - 5) + \log_9(x + 3) = 1$

(d) $7^{2x} = 2(7)^x + 3$ (e) $\sqrt[3]{\sqrt{x^5}} = 32$ (f) $7^{3x-2} = 5^{2x+1}$

(g) $3^{2x+1} - 10(3^x) + 3 = 0$ (h) $(\sqrt{27})^{5x+2}(9) = ({}^{1}\!/_{81})^{x+6}$

8. Rewrite ${}^{1}\!/_{12}$ and $\sqrt{2}\cdot 3^{-2}$ as $2^a \cdot 3^b$.

9. Sue invests \$100 paying 6% interest, compounded annually. John invests \$150, paying 4.5% compounded annually. Approximately when will their investments be worth the same?

10. The percentage, P_1, of caffeine remaining in your bloodstream is related to the elapsed time t in hours by $t = 5\left(\dfrac{\log P}{\log 0.5}\right)$.

(a) How long will it take for the amount of caffeine to drop to 35% of the amount consumed?

(b) What percentage of caffeine will remain $5\,{}^{1}\!/_{2}$ hour after drinking a cup of coffee?

11. An airplane altimeter is a gauge that indicates the height of the plane above the ground. The gauge works based on air pressure, according to the formula $h = 18400\log\dfrac{P_0}{P}$, where h is the height above the ground in metres, P is the air pressure at height h, and P_0 is the air pressure at ground level. Air pressure is measured in kilopascals (kPa).

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(a) Air pressure at the ground is 102 kPa. If the air pressure outside the plane is 32.5 kPa, what is the height of the airplane?

(b) How high would the airplane have to be flying for the outside air pressure at that height to be half the air pressure at ground level?

C. Unit 3 – Trigonometric Functions

Topics at a glance:

- special triangles & CAST rule
- graphs of the trigonometric functions & key features
- prove trigonometric identities
- trigonometric equations
- transformations & graphs of sinusoidal functions
- problems involving periodic behaviour & sinusoidal functions

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1. Determine the exact value of (a) $\dfrac{\sin\frac{4\pi}{3}\tan\frac{\pi}{6}}{\sec^3\frac{\pi}{4}}$ (b) $\dfrac{\left(\csc\frac{11\pi}{6}\right)\left(\cot\frac{\pi}{4}\right)}{\cos^3\left(\frac{5\pi}{3}\right)}$

5. Solve the equations on the interval $x \in [0, 2\pi]$:

(a) $3\csc^2 x = 5\csc x + 2$ (b) $5\sin x - 1 = 2\sin x + 1$

(c) $2\sin 2x = 1$ (d) $\left(\sqrt{3}\tan x - 1\right)\left(3\sin x - 2\right) = 0$

6. Prove the following: (a) $\csc 2x + \cot 2x = \cot x$ (b) $\dfrac{\cos x - \sec x}{\sin x - \sec x} = \cot x$

(e) $\dfrac{1+\tan x}{1+\cot x} = \dfrac{1-\tan x}{\cot x - 1}$ (f) $\sin\left(\frac{\pi}{4}+x\right)+\sin\left(\frac{\pi}{4}-x\right) = \sqrt{2}\cos x$

7. Given $\sec x = \frac{5}{4}$, $0 \le x \le \frac{\pi}{2}$ and $\sin y = \frac{12}{13}$, $\frac{\pi}{2} \le y \le \pi$, determine exact values for the following: (a) $\cos(x+y)$ (b) $\sin 2x$ (c) $\cot x - 2\csc^2 x$

8. (a) Find an exact value for $\sin\frac{\pi}{12}$.

(b) Rewrite $\sin 3\theta$ in terms of $\sin\theta$ only.

(c) Rewrite as a single trigonometric ratio and simplify completely:

(i) $8\cos^2 3x - 8\sin^2 3x$ (ii) $4\sin(2m)\cos(2m)$

(i) $8\cos^2 3x - 8\sin^2 3x$ (ii) $4\sin(2m)\cos(2m)$

9. (a) A sinusoidal function has a maximum value of –1, a minimum value of –9 and period $\pi/3$. Write the equation of (i) a sine function, and (ii) a cosine function, satisfying these conditions. Sketch the graph of the function.

(b) In Victoria, British Columbia, the first high tide was 3.99 m at 12:03 a.m. The first low tide of 0.55 m occurred at 6:24 a.m. The second high tide occurred at 12:19 p.m.

(i) Write a sinusoidal function to model the tides, using *t* to represent the number of hours, in decimals, since midnight.

(ii) Determine the height of the water at noon.

11. For each of the following: describe the transformations, find 2 *x*-intercepts without graphing (show your work), then sketch.

(a) $y = 1/4\left(3\tan\left(1/2(x-1) + \pi/3\right) + 2\right)$

(b) $y = \sqrt{5}(1/5\cos(4x - \pi/3) + 2)$

D. Unit 4 – Rate of Change & Combining Functions

Topics at a glance:
(a) Combining Functions:

- combining functions; key features
- composition of functions; key features
- composition of functions & transformations
- inverse of functions
- Solving equations & inequalities graphically & numerically
- function models

(b) Rates of Change:

- average rate of change; secant lines
- instantaneous rate of change; tangent lines
- solving problems involving rate of change
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1. Determine the average rate of change of each function on the specified interval:

(a) $y = \frac{5-3x}{x+1},\ 2 \le x \le 4$ (b) $y = 3\log(x+4) + 7,\ 0 \le x \le 5$

(c) $y = 2^{x+3} + 5,\ -2 \le x \le 1$ (d) $y = \sin 3x,\ 0 \le x \le \pi/4$

2. Find the instantaneous rate of change at the specified point:

(a) $y = x^3 - 5x^2 + 7,\ x = 1$ (b) $y = \frac{x+5}{x-2},\ x = 4$

3. (a) Determine the equation of the tangent line to $y = 5x^2 - 3x + 7$ at $x = 1$.

(b) The normal line is perpendicular to the tangent line and passes through the same point. What

3. (a) Determine the equation of the tangent line to

(b) The normal line is perpendicular to the tangent line and passes through the same point. What is the equation of the normal line to $y=5x^2-3x+7$ at $x=1$?

4. Water is draining from a large tank. After *t* minutes, there are $150{,}000-7500t+t^2$ litres of water in the tank

(a) Determine the average rate at which the water empties from the tank in the interval between 5 and 10 min.

(b) Determine the rate at which the water drains after exactly 8 min.

5. Given the functions $f(x)=x^2-5x$ and $g(x)=\frac{2-3x}{7}$, determine completely <u>simplified</u> expressions for each of the following:

(a) $f(x)g(x)$ (b) $f\left(g^{-1}(x)\right)$ (c) $f\left(f(x)\right)$ (d) $g\left(g(x)\right)$

6. $f(x)=3x-5$ and $g(x)=\frac{5}{x+4}$, find:

(a) $f\left(g(-1)\right)$ (b) $(f+g)(2)$ (c) $g\left(g(2)\right)$ (d) $f^{-1}\left(g(1)\right)$

7. Find the inverse relation:

(a) $y=\frac{\sqrt{3x-7}}{2}$ (b) $y=4^{2x}+3$

(c) $y=2x^2-5x+1$ (d) $y=\frac{1}{x+3}+2$

8. Given $f(x)=\frac{3x-1}{2x+5}$, find the inverse and show that $f^{-1}\left(f(x)\right)=x$.

9.

$f=\left\{(0,1),(1,2),(2,5),(3,10)\right\}$ and $g=\left\{(2,0),(3,1),(4,2),(5,3),(6,4)\right\}$.

Find: (i) $f^{-1}(x)$ (ii) $(f+g)(x)$ (iii) $f\left(g(x)\right)$

10. $f(x)$ is an even function, and $g(x)$ and $h(x)$ are both odd functions. State whether each of the following is even, odd or neither, and justify your answer (algebraic method required):

(a) $m(x)=h(x)g(x)$ (b) $k(x)=\frac{f(x)g(x)}{h(x)}$ (c) $n(x)=g(x)+h(x)$

11. In a vehicle test lab, the speed of a car *v* km/hr at *t* hours is represented by $v(t)=40+3t+t^2$. The rate of gasoline consumption of the car, c litres per km, at a speed *v* is represented by

11. In a vehicle test lab, the speed of a car v km/hr at t hours is represented by $v(t)=40+3t+t^2$. The rate of gasoline consumption of the car, c litres per km, at a speed v is represented by $c(v)=\left(\frac{v}{500}-0.1\right)^2+0.15$. Determine algebraically $c\left(v(t)\right)$, the rate of gasoline consumption as a function of time.

12. For each function h, find two functions f and g such that $h(x)=f\left(g(x)\right)$:

(a) $h(x)=\sqrt{2x^2+5}$ (b) $h(x)=5^{3x^2-7}$ (c) $h(x)=\sqrt[3]{\left(2x+4\right)^2}$ (d) $h(x)=\sin^2\left(5x+4\right)$

13. The Acme pen company estimates that the cost of manufacturing x pens is $C(x)=6000+0.8x$ and the revenue is $R(x)=\frac{1}{10,000}\left(30,000x-x^2\right)$.

(a) Sketch the graphs of $R(x)$ and $C(x)$ on the same axes.

(b) How many points of intersection does this system have? Explain their significance.

(c) Identify the region where $R(x)>C(x)$. Why is this region important?

(d) Use the superposition principle to graph the function $P(x)=R(x)-C(x)$.

(e) Use this function to determine the optimum number of units sold.