The Iterated Local Model for Social Networks

Erin Meger

Ryerson University
Joint work with:
Anthony Bonato, Huda Chuangpishit, Sean English, and Bill Kay

Combinatorics Seminar
CUNY Grad Center
15 November 2019

Outline

Complex Networks

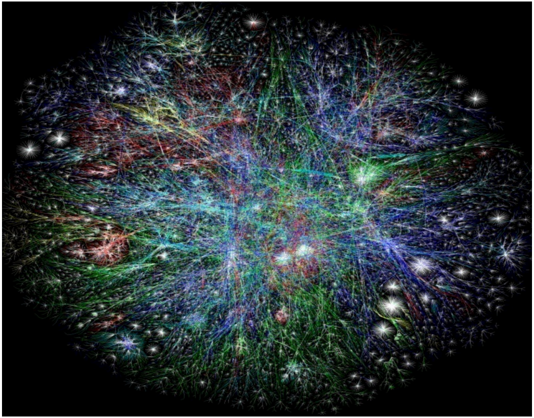


Figure: The Web Graph

Complex Networks

Four main properties:

1. Large-scale
2. Evolving over time
3. Power law degree distribution
4. Small world property

Complex Networks

Four main properties:

1. Large-scale
2. Evolving over time
3. Power law degree distribution
4. Small world property
5. Densification

Power Law Degree Distribution

Degree Distribution: $\{N_{k,G} : 0 \leq k \leq n\}$

$$N_{k,G} = |\{x \in V(G) : \deg_G(x) = k|$$

Power Law: for $1 < \beta \in \mathbb{R}$, and interval of $k \in \mathbb{N}$

$$\frac{N_{k,G}}{n} \approx k^{-\beta}.$$

Power Law Degree Distribution

Degree Distribution: $\{N_{k,G} : 0 \leq k \leq n\}$

$$N_{k,G} = |\{x \in V(G) : \deg_G(x) = k|$$

Power Law: for $1 < \beta \in \mathbb{R}$, and interval of $k \in \mathbb{N}$

$$\frac{N_{k,G}}{n} \approx k^{-\beta}.$$

Small World Property

The average distance is

$$L(G) = \frac{\sum_{u,v \in V(G)} d(u,v)}{\binom{|V(G)|}{2}}$$

The clustering coefficient of G is defined as follows:

$$C(G) = \frac{1}{|V(G)|} \sum_{x \in V(G)} C_x(G), \quad \text{where} \quad C_x(G) = \frac{\left|E\left(G[N_G(x)]\right)\right|}{\binom{\deg(x)}{2}}.$$

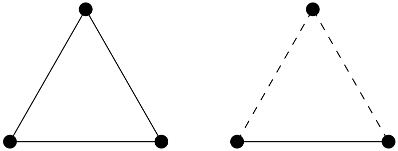
Densification

A sequence of graphs $\{G_t : t \in \mathbb{N}\}$ densifies over time if

$$\lim_{t \to \infty} \frac{|E(G_t)|}{|V(G_t)|} \to \infty$$

Structural Balance Theory

Representing adversarial relationships with $(-)$ and friendly relationships with $(+)$, Structural Balance Theory says triads seek a positive product of edge signs, called closure.



ILT

Iterated Local Transitivity Model (ILT) (2009, Bonato, Hadi, Horn, Prałat, Wang)

Input: G_0

To form G_t at time t clone each $x \in V(G_{t-1})$ by adding a new node x' such that

$$N_{G_t}(x') = N_{G_{t-1}}[x]$$

Example of ILT

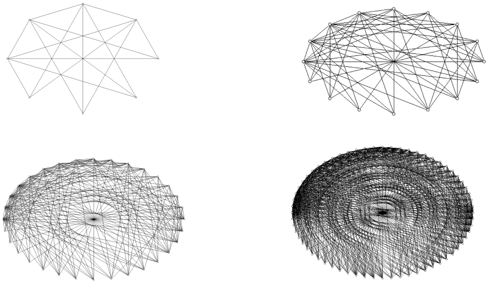


Figure: Example of ILT with $G_0 = C_4$.

Iterated Local Anti-Transitivity

Iterated Local Anti-Transitivity Model (ILAT) (2017, Bonato, Infeld, Pokhrel, Prałat)

Input: G_0

To form G_t at time t anti-clone each $x \in V(G_{t-1})$ by adding new node x^* such that

$$N_{G_t}(x^*) = V(G_{t-1}) \setminus N_{G_{t-1}}[x]$$

Example of ILAT

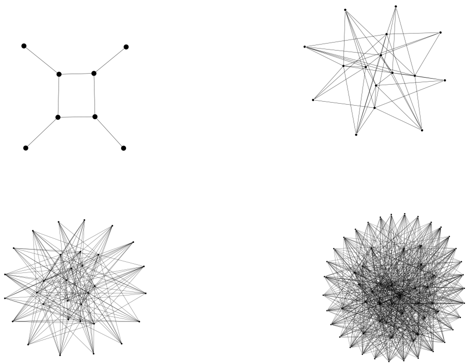


Figure: Example of ILAT with $G_0 = C_4$.

ILM

Iterated Local Model (ILM) (2019+, Bonato, Chuangpishit, English, Kay, M.)

Input: G_0 and $S = \{b_i\}_{i \in \mathbb{N}}$, where $b_i \in \{0, 1\}$

To form $\mathrm{ILM}_{t,S}(G_0)$ at time t:

- if $b_t = 1$ add a clone x' for each $x \in V(G_{t-1})$ with

$$N_{G_t}(x') = N_{G_{t-1}}[x]$$

- if $b_t = 0$ add an anti-clone x^* for each $x \in V(G_{t-1})$ with

$$N_{G_t}(x^*) = V(G_{t-1}) \backslash N_{G_{t-1}}[x]$$

Example of ILM

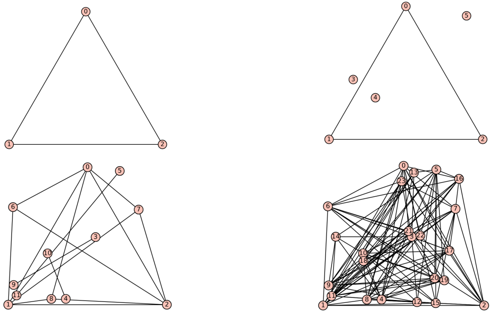


Figure: Example of ILM with $G_0 = K_3$ and $S = (0, 1, 0, 1, \ldots)$

Size, Evolution, and Densification

Theorem (2019+, BCEKM) Given any graph, G_0, and any binary sequence, S, with at least one zero, then at time step t

$$|E(\mathrm{ILM}_{t,S}(G))| = \Theta\left(2^{t+\beta}\left(\frac{3}{2}\right)^{t-\beta}\right) = \Theta\left(2^{\beta}\left(\frac{3}{2}\right)^{t-\beta} n_t\right)$$

Where τ is the first index such that $s_\tau = 0$, and β is the largest index such that $s_\beta = 0$.

Size, Evolution, and Densification

Consider the degrees of vertices in ILM

	$s_t = 0$	$s_t = 1$
$x \in V(G_{t-1})$	$n_t - 1$	$2\deg_{t-1}(x) + 1$
$x \notin V(G_{t-1})$	$n_{t-1} - \deg_{t-1}(x) - 1$	$\deg_{t-1}(x) + 1$

Size, Evolution and Densification

So far, ILM exhibits 3 of the 4+1 complex network properties

1. Large Scale
2. Evolving over time
5. Densification

Low Diameter

Theorem (2019+, BCEKM) Given $G \neq K_1$ be a graph that is not the disjoint union of two cliques, and a sequence with at least two zeroes, then

$$\mathrm{diam}(\mathrm{ILM}_{t,S}(G)) = 3$$

Low Diameter

Proof Sketch

- When G has domination at least 3, an Anti-Transitive step will ensure the diameter is less than 3 regardless of the starting diameter

Low Diameter

Proof Sketch

- When G has domination at least 3, an Anti-Transitive step will ensure the diameter is less than 3 regardless of the starting diameter
- Transitive steps do not change the diameter

Low Diameter

Proof Sketch

- When G has domination at least 3, an Anti-Transitive step will ensure the diameter is less than 3 regardless of the starting diameter
- Transitive steps do not change the diameter
- The graph is connected and has no dominating vertex

Clustering

Theorem (2019+, BCEKM) Given a sequence with bounded gaps between zeroes, and k a constant such that there are no gaps of length k,

$$C(\mathrm{ILM}_{t,S}(G)) \geq (1 + o(1))\frac{1}{2^{2k+4}}.$$

Clustering

Theorem (2019+, BCEKM) Given a sequence with bounded gaps between zeroes, and k a constant such that there are no gaps of length k,

$$C(\mathrm{ILM}_{t,S}(G)) \geq (1 + o(1))\frac{1}{2^{2k+4}}.$$

The clustering coefficient is bounded away from zero.

Small World

With non-zero clustering and low diameter, ILM exhibits its 4^{th} and final property

4. Small World Property

Induced Subgraphs

Theorem (2019+, BCEKM) If F is a graph, then there exists some constant $t_0 = t_0(F)$ such that for all $t \geq t_0$, all graphs G, and all binary sequences S, F is an induced subgraph of $\mathrm{ILM}_{t,S}(G)$.

Induced Subgraphs

Theorem (2019+, BCEKM) If F is a graph, then there exists some constant $t_0 = t_0(F)$ such that for all $t \geq t_0$, all graphs G, and all binary sequences S, F is an induced subgraph of $\mathrm{ILM}_{t,S}(G)$.

Proof by Picture :)

Hamiltonicity

Theorem (2019+, BCEKM)

Let $G \neq K_1$, and a binary sequence with at least two non-consecutive 0s, then for all $\text{ILM}_{t,S}(G)$ is Hamiltonian.

Other Structural Properties

For the model with certain restrictions on the input sequence and graph:

- $\chi(G) + t - 1 \leq \chi\left(\text{ILM}_{t,s}(G)\right) \leq \chi(G) + t$
- $\gamma(\text{ILM}_{t,s}(G)) \leq 3$

Future Directions

- Graph Limits
- Domination number in remaining cases
- Randomization of the model
- Improve Clustering, still unknown for ILAT

Thank You