Cops, Robbers, and Barricades

A new variant of a classic game

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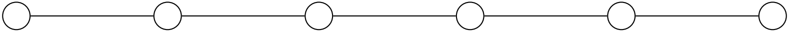
Introduction to Cops and Robbers

Definition

Cops and Robbers is a game on a graph, G, with two players, the set of cops, C and the single robber, R, the game plays as follows:

- C play on any vertices of G
- R plays on any unoccupied vertex of G
- Players alternate turns moving along edges
- C win by landing on the vertex occupied by R, *capture*
- R wins by evading capture indefinitely

Example of the Game



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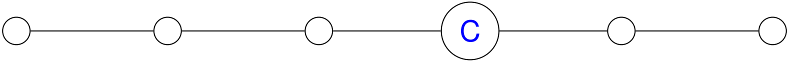

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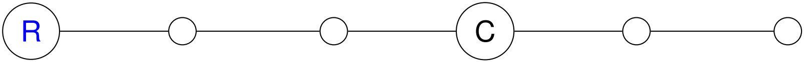

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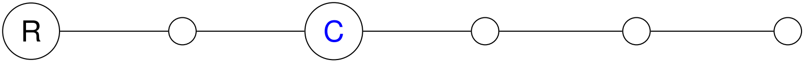

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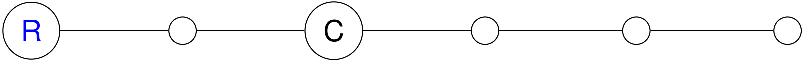

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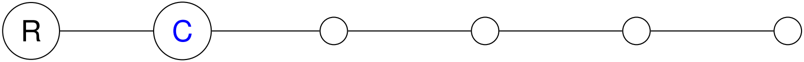

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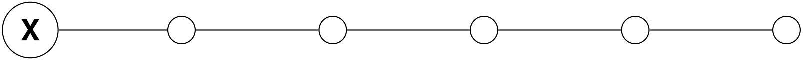

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corner: u is a corner if there exists some v such that $N[u] \subseteq N[v]$

Theorem (N+W [6])

A graph is copwin if and only if there is some sequence of deleting corners that results in a single vertex

Nowakowski Winkler Characterization

For any graph, G, we define a binary relation $\leq_\alpha$, for each $\alpha \in \mathbb{N}$, on the set of vertices of G.

- $x \leq_0 y \iff x = y$
- $x \leq_\alpha y \iff$ for all $u \in N(x)$ there is some $v \in N(y)$ such that $u \leq_p v$ with $p < \alpha$.

Theorem (N+W[6])

G is copwin if and only if $x \leq y$ for all $x, y \in V(G)$.

Meyniel's Conjecture

For some integer n, let $c(n)$ be the maximum cop number $c(G)$ over all graphs G on n nodes.

Conjecture (Meyniel's Conjecture)

$$c(n) = O(\sqrt{n})$$

Other Bounds

Asymptotic Bounds:

- Frankl $c(n) \leq (1 + o(1))n\frac{\log \log n}{\log n}$
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Conjecture (Soft Meyniel Conjecture)

For a fixed constant $c > 0$

$$c(n) = O(n^{1-c})$$

Variants

- Original Game [Quilliot '78, Nowakowski and Winkler '83]
- *k*-cop-win [Aigner and Fromme '84]

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- *k*-cop-win [Aigner and Fromme '84]
- Variants:
  - tandem-win
  - photo-radar
  - lazy cops
  - distance *k*
  - zombies
  - decoy
  -
  -

Cops, Robbers and Barricades

Definition

Cops, Robbers, and Barricades is a game on a graph, G, with two players, the set of cops, C and the single robber, R, the game plays as follows:

- C play on any vertices of G
- R plays on any unoccupied vertex of G
- R may build a barricade on an unoccupied vertex. Barricades block all players from occupying that vertex.
- Players alternate turns moving along edges
- C win by landing on the vertex occupied by R, *capture*
- R wins by evading capture indefinitely

Cops, Robbers, and Barricades

Give power to the Robber:

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Generalize: Robber can build *b* barricades at ‘any’ location.
Barricade Rules: permissible locations for the barricades.

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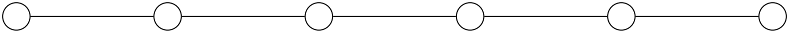

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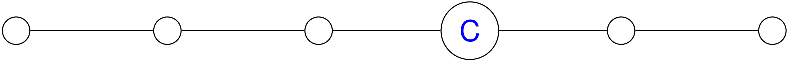

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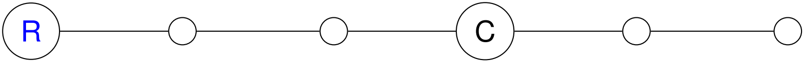

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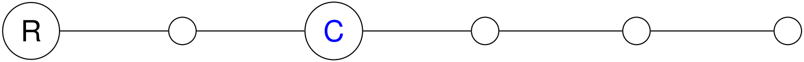

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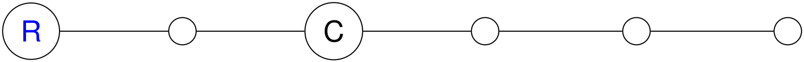

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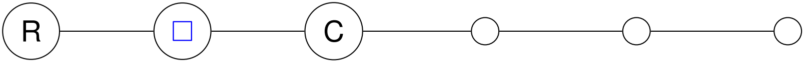

Barricade-Cop Number

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Theorem

For any graph G, $c(G) \leq c_B(G)$.

Results

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For a path with n nodes, P_n, $c_B(P_n) = \lceil \frac{(n-2)}{7} \rceil$.

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Theorem

For any tree, T, let ℓ be the set of leaves of T, then $c_B(T) \geq |N_2(\ell)|$.

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A tree with a root, and having each branch of length 1 or 2, called a spider, is barricade-cop-win.

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Theorem

The only trees that are barricade-cop-win are spiders.

M-ary Trees

Definition

A tree is called *m*-ary if each vertex has at most m children.

A tree is said to be full *m*-ary, if every non-leaf has exactly m children.

The height of a tree, h, is the number of edges in a longest path from the root to any leaf.

M-ary Trees

Theorem

For a full m-ary tree, T with height h, we let $c \leq \lceil \frac{h-m-3}{2m+3} \rceil$*, then,*

$$c_B(T) \leq \sum_{i=1}^{c} m^{i(2m+4)+m+1}$$

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Place a row full of cops at height $h-2$ (ie. the grandparents of the leaves), and then place full rows of cops at every $2m+3$ rows, until we are at height $m+1$ or less. This is called the row strategy.

M-ary Trees

Distance Strategey: Place each cop so that they are no closer than 4 vertices away from any other

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Conjecture

For a tree, T,

$$c_B(T) = \max \left(\lceil \frac{n}{4} \rceil, \sum_{i=1}^{c} m^{i(2m+4)+m+1} \right)$$

Diameter

Definition

The diameter of a graph G, is the maximum length shortest path between any two vertices, $x, y \in V(G)$.

Theorem

If G has diameter 2, $c_B(G) = c(G)$.

Domination

Is this graph parameter similar to domination number?

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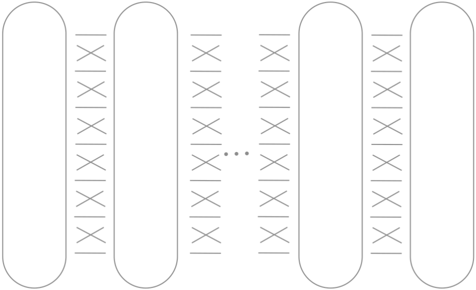

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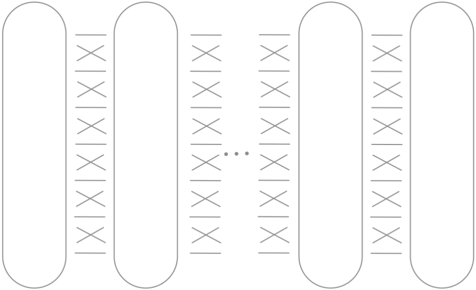

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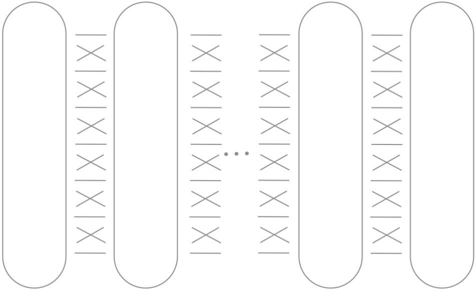

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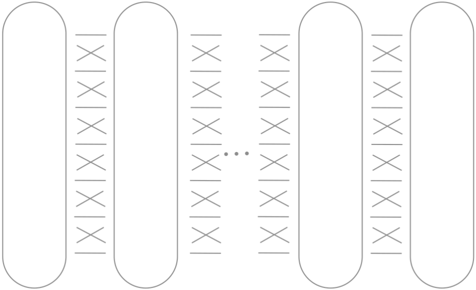

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$m \geq \lfloor \frac{d}{2} \rfloor$

$c_B(G) = 2$ but $\gamma(G) \to \infty$

The Characterization

Definition (Some Notation)

- Barricade Set B_k: the set of vertices containing all barricades at round k
- Barricade Sequence: the sequence $\{B_1, B_2, ..., B_k\}$
- Barricade Graph G_k: the graph G with B_k removed
- $comp(R)$: the component of G_k containing R

The Characterization

Using the relation, $\leq$ as defined in the Nowakowski, Winkler characterization.

Theorem

A graph G is barricade-cop-win if and only if given any sequence of B_k, there is some vertex $u \in N_{G_{k-1}}(C)$ with $u \in comp_{G_k}(R)$, such that $u \leq_\omega R$ for some finite ω.

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This gives us an algorithm for determining whether a graph is barricade cop win. Although it is not in poly-time like the Nowakowski Winkler algorithm. [6] [4] [5]

Thank You

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